Definition:

The "space of conformal blocks" $\mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$ is defined as the space of linear maps

$$\Psi : H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n} \longrightarrow \mathbb{C}$$

which are invariant under diagonal action Δ of the Lie algebra $\mathfrak{g}(p_1, \dots, p_n)$, i.e.

$$\sum_{j=1}^{n} \Psi(\xi_1, \dots, (X \otimes f_{p_j})\xi_j, \dots, \xi_n) = 0$$

$\forall\ \xi_1 \in H_{\lambda_1}, \dots, \xi_n \in H_{\lambda_n}$ and $X \otimes f \in \mathfrak{g}(p_1, \dots, p_n)$

Notation: $\mathrm{Hom}_{\mathfrak{g}(p_1, \dots, p_n)}\left(\bigotimes_{j=1}^{n} H_{\lambda_j}, \mathbb{C}\right)$

Consider the meromorphic function

$$f(z) = (z - z_i)^r, \quad r < 0$$

defined on $\mathbb{CP}^1$. Taylor expansion at p_j:

$$f_{p_j}(t_j) = \sum_{m=0}^{\infty} a_m^{(j)} t_j^m$$

Then invariance property of Ψ gives:

$$\Psi(\xi_1, \dots, (X \otimes t_i^r)\xi_i, \dots, \xi_n) \qquad (*)$$

$$= - \sum_{j; j \neq i} \sum_{m \geq 0} a_m^{(j)} \Psi(\xi_1, \dots, (X \otimes t_j^m)\xi_j, \dots, \xi_n)$$

$\forall\ \xi_j \in H_j, \quad 1 \le j \le n.$

Define the embedding map:

$$\iota : \bigotimes_{j=1}^{n} V_{\lambda_j} \longrightarrow \bigotimes_{j=1}^{n} H_{\lambda_j}$$

finite dim. irreducible rep. of $\mathfrak{g}$ with highest weight λ_j

$\longrightarrow$ restriction map:

$$\iota^* \Psi : \bigotimes_{j=1}^{n} V_{\lambda_j} \longrightarrow \mathbb{C}$$

where $\iota^*\Psi = \Psi \circ \iota$. Then we have the following

<u>Lemma</u>:

For $\Psi \in \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n)$, its restriction Ψ_0 on $\bigotimes_{j=1}^{n} V_{\lambda_j}$ is invariant under $\Delta(\mathfrak{g})$.

Moreover,

$$\iota^* : \mathcal{H}(p_1, \dots, p_n; \lambda_1, \dots, \lambda_n) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}\left(\bigotimes_{j=1}^{n} V_j, \mathbb{C}\right)$$

is injective. $\longrightarrow$ Ψ is uniquely determined by Ψ_0.

<u>Proof</u>:

Ψ is in particular invariant under diagonal action of $X \otimes 1_{\mathbb{CP}^1}$

$\in \mathfrak{g}$ constant function on $\mathbb{CP}^1$

$\longrightarrow$ for any $X \in \mathfrak{g}$, $\xi_j \in V_{\lambda_j}$, $1 \le j \le n$:

$$\sum_{j=1}^{n} \Psi(\xi_1 \otimes \cdots \otimes X\xi_j \otimes \cdots \otimes \xi_n) = 0$$

Now set

$$\mathcal{F}_d = \bigoplus_{d_1+\cdots+d_n \le d} \left(\bigotimes_{j=1}^{n} \mathcal{H}_{\lambda_j}(d_j) \right)$$

using the direct sum decomposition

$\mathcal{H}_{\lambda_j} = \bigoplus_{d \ge 0} \mathcal{H}_{\lambda_j}(d)$, Recall: $\mathcal{H}_\lambda(d)$ is eigenspace of L_0 with eigenvalue $\Delta_\lambda + d$.

We have $\mathcal{F}_0 = \bigotimes_{j=1}^{n} V_{\lambda_j}$ and

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_d \subset \cdots$$

Suppose $\Psi\big|_{\mathcal{F}_0} = 0$. Have to show $\Psi(\xi) = 0$ where

$$\xi = \eta_1 \otimes \cdots \otimes (X \otimes t_i^r)\eta_i \otimes \cdots \otimes \eta_n,$$

$$X \in \mathfrak{g},\ r < 0,\ \eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{F}_{d-1}$$

$\longrightarrow$ proof follows by induction on d by using identity $(*)$. $\square$

Recall:

H_λ is quotient of M_λ by submodule generated by $\chi = (E \otimes t^{-1})^d v, \quad d = k - \lambda + 1, \quad v \in V_\lambda$

null-vector

Set $d_i = k - \lambda_i + 1, \quad 1 \le i \le n$. Then we have

Proposition 1:

For Ψ belonging to above space of conformal blocks, the restriction map

$$\Psi_0 : V_{\lambda_1} \times \cdots \times V_{\lambda_n} \longrightarrow \mathbb{C}$$

satisfies

$$\Psi_0 (E^{m_1}\xi_1, \ldots, E^{m_{i-1}}\xi_{i-1}, v_i, E^{m_{i+1}}\xi_{i+1}, \ldots, E^{m_n}\xi_n) = 0$$

for v_i highest weight vector, $\xi_j \in V_j$, $j \ne i$, $1 \le j \le n$, and $m_j \geq 0$, $1 \le j \le n$, $\sum_{j, j \ne i} m_j = d_i$

Proof:

We show the statement in the case $i = 1$.

$\chi_1 = (E \otimes t_1^{-1})^{d_1} v_1$ null-vector

$$\longrightarrow \Psi((E \otimes t_1^{-1})^{d_1} v_1, \xi_2, \ldots, \xi_n) = 0$$

Applying equation $(*)$ then gives:

$$\sum_{m_2+\cdots+m_n=d_1} \frac{d_1!}{m_2!\cdots m_n!} \prod_{2\leq j\leq n} (z_j - z_1)^{-m_j} f_{m_2,\dots,m_n} = 0$$

where

$$f_{m_2,\dots,m_n} = \Psi_0\left(v_1, E^{m_2}\xi_2, \dots, E^{m_n}\xi_n\right) \quad (**)$$

To see this, consider $d_1 = 1$

$$t_1^{-1} = (z - z_1)^{-1}$$

$$\Rightarrow t_1^{-1} = (t_2 + z_2 - z_1)^{-1} = (z_2 - z_1)^{-1} - \frac{t_2}{(z_2-z_1)^2} + \frac{t_2^2}{(z_2-z_1)^3} + O(t_2^3)$$

$$\overset{(*)}{\Longrightarrow} \Psi((E\otimes t_1^{-1})v_1, \xi_2, \dots, \xi_n)$$

$$= -\sum_{j, j\neq 1} \sum_{m\geq 0} (z_j - z_1)^{-m-1} (-1)^m \Psi(v_1, \dots, (E\otimes t_j^m)\xi_j, \dots, \xi_n)$$

$$= -\sum_{j, j\neq 1} (z_j - z_1)^{-1} \Psi(v_1, \dots, E\xi_j, \dots, \xi_n)$$

$\longrightarrow$ general case follows from induction

Since $(**)$ holds for any $z_1, \dots, z_n$

$\longrightarrow f_{m_2,\dots,m_n} = 0$ $\square$

Denote by $N_{\lambda_1\lambda_2\lambda_3}$ the dimension of the space of conformal blocks $\mathcal{H}(p_1,p_2,p_3;\lambda_1,\lambda_2,\lambda_3)$

Then we have

Proposition 2:

In the case $n=3$, if the following holds

$$\lambda_1+\lambda_2+\lambda_3 \in 2\mathbb{Z},$$
$$|\lambda_1-\lambda_2| \leq \lambda_3 \leq \lambda_1+\lambda_2,$$
$$\lambda_1+\lambda_2+\lambda_3 \leq 2k,$$

then $N_{\lambda_1\lambda_2\lambda_3}=1$. Otherwise, $N_{\lambda_1\lambda_2\lambda_3}=0$

$\longrightarrow$ "quantum Clebsch-Gordan rule" at level k.

Proof:

It is known that (Clebsch-Gordan rule)

$$\mathrm{Hom}_{sl_2(\mathbb{C})}(V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}, \mathbb{C}) \cong \mathbb{C}$$

holds if and only if the condition

$$\lambda_1+\lambda_2+\lambda_3 \in 2\mathbb{Z},$$
$$|\lambda_1-\lambda_2| \leq \lambda_3 \leq \lambda_1+\lambda_2$$

is satisfied. If the Clebsch-Gordan condition is not satisfied, we have:

$$\mathrm{Hom}_{sl_2(\mathbb{C})}(V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}, \mathbb{C}) = 0$$

Have already shown (see above Lemma):

$$\mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3) \subset \mathrm{Hom}_{sl_2(\mathbb{C})}(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3}, \mathbb{C})$$

as a subspace.

$\longrightarrow$ $N_{\lambda_1 \lambda_2 \lambda_3}$ is either 0 or 1.

We show:

$$N_{\lambda_1 \lambda_2 \lambda_3} = 1 \iff \lambda_1 + \lambda_2 + \lambda_3 \leq 2k \text{ in addition to Clebsch-Gordan cond.}$$

We take H, E and F as a basis of $sl_2(\mathbb{C})$. For $\Psi \in \mathcal{H}(p_1, p_2, p_3; \lambda_1, \lambda_2, \lambda_3)$ we then have

$$\text{Proposition 1} \implies \Psi_0(v_1, E^{m_2}\xi_2, E^{m_3}\xi_3) = 0, \quad (*)$$

$$m_2 + m_3 = k - \lambda_1 + 1$$

for highest weight vector $v_1 \in V_{\lambda_1}$ and any $\xi_j \in V_{\lambda_j}$, $j = 2, 3$. Take ξ_2 and ξ_3 to be eigenvectors of H with eigenvalues α_2, α_3.

$$\longrightarrow -\lambda_j \leq \alpha_j \leq \lambda_j, \quad \alpha_j = -\lambda_j + 2n_j, \quad n_j \in \mathbb{Z}_+$$

We have

$$H(v_1 \otimes E^{m_2}\xi_2 \otimes E^{m_3}\xi_3)$$

$$= (\lambda_1 + 2(k - \lambda_1 + 1) - \lambda_2 + 2n_2 - \lambda_3 + 2n_3)\, v_1 \otimes E^{m_2}\xi_2 \otimes E^{m_3}\xi_3$$

$$= (2(k+1) - (\lambda_1 + \lambda_2 + \lambda_3) + 2(n_2 + n_3))\, v_1 \otimes E^{m_2}\xi_2 \otimes E^{m_3}\xi_3$$

If $\lambda_1+\lambda_2+\lambda_3 \leq 2k$, then we have

$$2(k+1)-(\lambda_1+\lambda_2+\lambda_3)+2(n_2+n_3) \neq 0$$

and $(*)$ follows from invariance of Ψ_0 under diagonal action of H.

Consider now the case $\lambda_1+\lambda_2+\lambda_3 > 2k$.

We want to show that $\Psi_0(\eta_1,\eta_2,\eta_3)=0$ for any $\eta_j \in V_{\lambda_j}$. Suppose that

$$H(\eta_1\otimes\eta_2\otimes\eta_3)=0 \quad \left(\begin{array}{l}\text{otherewise inv. under} \\ H \text{ implies } \Psi_0(\eta_1,\eta_2,\eta_3)=0\end{array}\right)$$

and take $\eta_1 = v_1$ highest weight.

Set $H\eta_2 = -\lambda_2\eta_2$, and $H\eta_3 = (\lambda_2-\lambda_1)\eta_3$.

Then $\lambda_1+\lambda_2+\lambda_3 > 2k$ implies

$$\begin{aligned}\lambda_2-\lambda_1-2d_1 &= \lambda_2-\lambda_1-2(k-\lambda_1+1) \\ &= \lambda_1+\lambda_2-2(k+1) \geq -\lambda_3\end{aligned}$$

$\Rightarrow \eta_3 = E^{d_1}\xi$ for some $\xi \in V_{\lambda_3}$.

But then $(*) \Rightarrow \Psi_0(v_1,\eta_2,\eta_3)=0$

Using $\mathfrak{g}$-invariance of Ψ_0 then shows by induction that Ψ_0 vanishes on $V_{\lambda_1}\otimes V_{\lambda_2}\otimes V_{\lambda_3}$.

Hence $N_{\lambda_1\lambda_2\lambda_3}=0$.

$\square$